A Metric Theory of Gravitation without Minimal Coupling of Matter and Gravitational Field

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A metric theory of gravitation is suggested which reduces to Einstein's theory in the case of vanishing matter. If matter is present, in the Lagrangian formulation of the theory the principle of minimal coupling is given up by directly linking the matter variables to the curvature tensor. The theory contains a free parameter of dimension length. It is considered not to be a universal constant but a length characteristic for the mass of the material system described.

Results diverging from those of General Relativity are to be expected for regions with high curvature i. e. especially for gravitational collapse and dense phases of the cosmos. An exact, static and spherically symmetric solution with constant matter density is discussed; it indicates that, possibly, gravitational collapse is avoided.

1. Introduction

Einstein's theory of general relativity represents both a theory of gravitation and a framework for macroscopic theories of matter. In the Lagrangian formulation the field equations derive from a variational principle with scalar Lagrangian density:

$$\mathcal{L} = \mathcal{L}_F + 2\,\varkappa\,\mathcal{L}_M \tag{1}$$

where $\varkappa = 8\,\pi\,G/c^4$ is the coupling constant and *

$$\mathcal{L}_F = \sqrt{-g}\,R^\sigma{}_\sigma \tag{2}$$

describes the gravitational field while

$$\mathcal{L}_M = \sqrt{-g}\,L_M\,(g_{\varkappa\lambda}, u^A, u^A{}_{;\alpha,\dots}) \tag{3}$$

is the matter part of the Lagrangian. u^A denotes the variables of matter and matter fields (A is a global index). By variation of the action integral with regard to the metric and the variables u^A the field equations of gravitation and of matter follow.

The important point, here, is the prescription for obtaining L_M from the Lagrangian of the corresponding material system in *special relativity:* Replace, in the special relativistic Lagrangian $L_M(\eta_{\varkappa\lambda}, u^A, u^A{}_{,\alpha,\dots})$ the Minkowski metric $\eta_{\varkappa\lambda}$ by the Riemannian metric $g_{\varkappa\lambda}$ and the partial derivative $\partial/\partial x^\alpha$ by the covariant derivative with regard to $g_{\varkappa\lambda}$. This prescription guarantees that, in each event, a coordinate system (geodesic coordinates) can be found in which the general relativistic theory reduces to the expressions of special relativity. This procedure is considered an expression of the *principle of minimal coupling* [2]. The gravitational field described by $g_{\varkappa\lambda}$ is coupled to the matter variables u^A via $g_{\varkappa\lambda}$ and $g_{\varkappa\lambda,\nu}$ only, not via second derivatives.

In the opinion of many relativists the principle of minimal coupling is a necessary consequence of the strong equivalence principle by the following line of reasoning. The strong equivalence principle states that, in a local inertial system, the physical laws hold as formulated in special relativity theory. The local inertial system is identified with the geodesic coordinates. As far as the equivalence principle is supported by, for example, the free fall experiments of Eötvös, Renner, Dicke and Braginsky [3, 4] the principle of minimal coupling is a sine qua non condition for any theory describing the interaction of matter and gravitation.

As direct consequence of minimal coupling we may take the equation for the matter tensor $T^{\alpha\beta}$:

$$T^{\alpha\beta}{}_{;\beta} = 0 \tag{4}$$

following from Einsteins field equations

$$G^{\alpha\beta} = -\varkappa\,T^{\alpha\beta} \tag{5}$$

and the contracted Bianchi identities. In a local inertial system, it coincides with the special relativistic conservation laws for energy and momentum $T^{\alpha\beta}{}_{,\beta} = 0$. Replacements of the type suggested, for example, by Rastall [5] of Eq. (4)

$$T^{\alpha\beta}{}_{;\beta} = \lambda\,R^\sigma{}_{\sigma;\beta}\,g^{\alpha\beta} \tag{6}$$

violate the formulation of the equivalence principle given above [6].

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* Greek indices run from 0 to 3, Latin indices from 1 to 3. Sign $(g_{\varkappa\lambda}) = -2$. For the definition of $R^\alpha{}_{\beta\gamma\delta}$ and $R_\sigma{}^\sigma$ the sign conventions of Eisenhart [1] are used.

Nevertheless, some doubts seem to be permitted as to this line of arguments. The high-precision measurements backing the equivalence principle have all been made *outside* of matter. In fact, with a few exceptions [4, 7, 8], it is rarely touched upon what meaning Einstein's thought experiment with the freely falling elevator has *within macroscopic* matter. Also, in the derivation of Eq. (4) from kinetic theory [9] [without use of the field equations (5)] it is tacitly assumed that the principle of minimal coupling holds.

In view of the many theories of gravitation devised which contain additional geometrical objects (scalar fields, tetrads etc.) or even leave the framework of Riemannian geometry, it many be worth to investigate theories of the type of General Relativity in which the principle of minimal coupling is given up, provided that those results of Einstein's theory supported strongly by observation and experiment are regained.

A class of such theories is suggested in Sect. 2 and narrowed to one single theory whose field equations are then derived. In Sect. 3 the case of an ideal fluid matter distribution is studied. In Sect. 4 the point particle approach to matter is dealt with, briefly. The Newtonian approximation of the theory is treated in Sect. 5 while Sect. 6 is devoted to spherically symmetric, static gravitational fields. An exact solution for constant energy density is discussed. In a concluding section we give an outlook on further possibilities of breaking the principle of minimal coupling.

2. A Theory of Gravitation without Minimal Coupling

We set the following guideposts for the construction of a theory without minimal coupling:

1) The field equations ought to be second order partial differential equations for $g_{\alpha\beta}$ and to follow from a variational principle.

2) The variables of matter (matter fields) are coupled directly to the curvature tensor. This is to be done in a way independent of any specific material system (universal coupling).

3) If $L_M = 0$, i. e. if no matter is present, the field equations ought to coincide with the Einstein vacuum field equations.

4) If $R^\alpha{}_{\beta\gamma\delta} = 0$, i. e. if no permanent gravitational fields are present, then, in each event, the special relativistic conservation law $T^{\alpha\beta}{}_{,\beta} = 0$ ought to follow.

Of course, these assumptions cannot determine the Lagrangian of the theory. Detailed investigations will have to show which if any of the various possibilities is most preferable. We now replace the Lagrangian density (1) by

$$\mathcal{L} = \mathcal{L}_F + 2\varkappa A(R^\alpha{}_{\beta\gamma\delta})\,\mathcal{L}_M \tag{7}$$

where, as in Eqs. (2), (3)

$$\mathcal{L}_F = \sqrt{-g}\,R^\sigma{}_\sigma \equiv \sqrt{-g}\,R$$

$$\mathcal{L}_M = \sqrt{-g}\,L_M(g_{\varkappa\lambda}, u^A, u^A{}_{;\alpha}, \dots)$$

while $A(R^\alpha{}_{\beta\gamma\delta})$ is a scalar function of the (fourteen) scalar invariants of the curvature tensor satisfying

$$A(0) = 1\,. \tag{7}$$

A simple choice for A makes it a function of the Ricci scalar R alone:

$$A(R^\alpha{}_{\beta\gamma\delta}) = A(R)\,. \tag{8}$$

The variation of the Lagrangian (7) with respect to $g_{\alpha\beta}$ leads to

$$\delta\mathcal{L} = \delta\mathcal{L}_F + 2\,\varkappa\,(\delta A\,\mathcal{L}_M + A\,\delta\mathcal{L}_M)$$

where, with Eq. (8)

$$\mathcal{L}_M\,\delta A = \sqrt{-g}\,[-A'\,L_M\,R^{\alpha\beta} + (A'\,L_M)_{;\varkappa;\lambda}(g^{\varkappa\lambda}g^{\alpha\beta} - g^{\varkappa\alpha}g^{\lambda\beta})]\,\delta g_{\alpha\beta} + \text{divergence terms} \tag{9}$$

and $A' := \mathrm{d}A/\mathrm{d}R$. Leaving aside the divergence terms, the field equations become:

$$G^{\alpha\beta} = -\varkappa\,[A\,T^{\alpha\beta} + 2\,A'\,L_M\,R^{\alpha\beta} - 2\,(A'\,L_M)_{;\varkappa;\lambda}(g^{\varkappa\lambda}g^{\alpha\beta} - g^{\varkappa\alpha}g^{\lambda\beta})]\,. \tag{10}$$

Here, $T^{\alpha\beta} := (-2/\sqrt{-g})\,\delta\mathcal{L}_M/\delta g_{\alpha\beta}$ is the energy-momentum tensor of Einstein's theory.

In order to obtain second order equations we put

$$A = 1 + \varepsilon\,l_0{}^2 R\,, \quad \varepsilon = \pm 1\,; \tag{8}$$

l_0 is a constant of dimension length which we interpret to characterize the mass of the matter distribution (see below and Section 5). The sign factor ε is left undetermined, at this stage. To this extent l_0 cannot be fitted arbitrarily to observational data.

With (8) we arrive at the field equations

$$G^{\alpha\beta} = -\varkappa\,\Theta^{\alpha\beta} \tag{11}$$

with the new energy-momentum tensor of matter:

$$\Theta^{\alpha\beta} := (1 + \varepsilon\, l_0^2 R) T^{\alpha\beta} + 2\,\varepsilon\, l_0^2 L_M R^{\alpha\beta} - 2\,\varepsilon\, l_0^2 L_{M;\varkappa;\lambda} (g^{\varkappa\lambda} g^{\alpha\beta} - g^{\varkappa\alpha} g^{\lambda\beta}) \,. \tag{12}$$

For notational convenience we introduce $\lambda := \varepsilon\, l_0^2\, \varkappa$.

From Eq. (11)

$$\Theta^{\alpha\beta}{}_{;\beta} = 0 \tag{13}$$

which, if expressed in terms of the energy-momentum tensor and the matter Lagrangian of Einstein's theory, reads as

$$T^{\alpha\beta}{}_{;\beta} = -[\log(1 + \varepsilon\, l_0^2 R)]_{,\beta} T^{\alpha\beta} - L_M [\log(1 + \varepsilon\, l_0^2 R)]_{,\beta}\, g^{\alpha\beta} \,. \tag{14}$$

There is no unusual ambiguity in this definition of the new energy-momentum tensor $\Theta^{\alpha\beta}$. The obvious choice is to take the r.h.s. of the field equations which directly results from the variation

$$(-2/\sqrt{-g})\, \delta(\mathcal{L} - \mathcal{L}_F)/\delta g_{\alpha\beta}$$

except for the dropping of divergence terms. However, the name may appear strange as $\Theta^{\alpha\beta}$ now contains the Ricci tensor explicitely. This reflects the violation of minimal coupling. In the present theory of gravitation it is even more difficult to separate out material sources and gravitational field than in General Relativity.

In a local inertial system, $\Theta^{\alpha\beta}$ does not reduce to the special relativistic energy-momentum tensor of matter. On the other hand one can remove the Ricci tensor from the r.h.s. of the field equations (11) by contracting them, i. e. with the help of

$$R = (1 - \lambda\, T^\sigma{}_\sigma - 2\,\lambda\, L_M)^{-1} [\varkappa\, T^\sigma{}_\sigma - 6\,\lambda\, L_{M,}{}^\sigma{}_{;\sigma}] \,. \tag{15}$$

The resulting expression is:

$$\begin{aligned} \Theta^{\alpha\beta} = {} & (1 + 2\,\lambda\, L_M)^{-1} \\ & \cdot (1 - 2\,\lambda\, L_M - \lambda\, T^\sigma{}_\sigma)^{-1} \{ T^{\alpha\beta} [-\varkappa (1 - 2\,\lambda\, L_M) \\ & + 6\,\lambda^2 L_{M,}{}^\sigma{}_{;\sigma}] - \varkappa\,\lambda\, L_M T^\sigma{}_\sigma\, g^{\alpha\beta} \\ & + 2\,\lambda\,(1 + \lambda\, L_M - \lambda\, T^\sigma{}_\sigma) L_{M,}{}^\nu{}_{;\nu}\, g^{\alpha\beta} \} \\ & - 2\,\lambda\,(1 + 2\,\lambda\, L_M)^{-1} L_{M,}{}^{\alpha}{}_{;}{}^{\beta} \,. \end{aligned} \tag{12'}$$

In the limit $\lambda = 0$ the new theory coincides with Einstein's theory of gravitation. $\lambda = 0$ means that matter does not gravitate but is considered in the test mass limit, only.

From Eq. (8) it is clear that the results of the new theory will not diverge much from those of Einstein's if $|\varepsilon\, l_0^2 R^\sigma{}_\sigma| \ll 1$. For an estimate we use the Schwarzschild vacuum solution (which is an exact solution for both theories) to obtain $l_0^2\, r_s \ll a^3$ where a is the radius and $r_s = 2\,G M c^{-1}$ the Schwarzschild radius of the gravitating object. We now put:

$$l_0 = r_s \,. \tag{16}$$

Thus, for ordinary stars and white dwarfs, but not for neutron stars and collapsing objects the results of the new theory should be the same as in General Relativity.

3. Gravitating Ideal Fluid

In order to apply the preceeding theory we start from the special relativistic Lagrangian for an ideal fluid. The fluid is described by its rest mass density ϱ, its internal energy density $e(\varrho)$, its total energy density $F(\varrho)$ with $F = \varrho[c^2 + e(\varrho)] =: \varrho\, y$ and its 4-velocity u^α. Upon these matter variables the following constraints are imposed

$$c^2 = \eta_{\varkappa\lambda}\, u^\varkappa u^\lambda \,, \tag{17 a}$$

$$0 = (\varrho\, u^\varkappa)_{,\varkappa} \,. \tag{17 b}$$

With Ray [10] we add the constraint equations to the matter Lagrangian:

$$L_M = -F \tag{18}$$

by means of Lagrangian multipliers λ_1, λ_2:

$$L_M + \lambda_1 (\eta_{\varkappa\lambda}\, u^\varkappa u^\lambda - c^2) + \lambda_2 (\varrho\, u^\varkappa)_{,\varkappa} \,.$$

Transition to the Lagrangian of the new theory, according to Eq. (7) leads to

$$\mathcal{L} = \sqrt{-g} [R^\sigma{}_\sigma - 2\,\varkappa (1 + \varepsilon\, l_0^2 R^\sigma{}_\sigma) F + \lambda_1 (g_{\varkappa\lambda}\, u^\varkappa u^\lambda - c^2) + \lambda_2 (\varrho\, u^\varkappa)_{;\varkappa}] \,. \tag{19}$$

Direct variation with regard to $g_{\alpha\beta}$, u^α, ϱ, λ_1 and λ_2 and elimination of the multipliers brings back the constraint

$$g_{\alpha\beta}\, u^\alpha u^\beta = c^2 \tag{20 a}$$

the conservation of rest mass

$$(\varrho\, u^\alpha)_{;\alpha} = 0 \tag{20 b}$$

the equations of motion:

$$\dot{u}^\alpha - c^2 h_\sigma{}^\alpha\, [\log F'(1 + \varepsilon\, l_0^2 R)]_,{}^\sigma = 0 \tag{21}$$

where $F' = \mathrm{d}F/\mathrm{d}\varrho$ and the field equations

$$G^{\alpha\beta} = -\varkappa [\mu\, u^\alpha u^\beta + u^{(\alpha} q^{\beta)} - \tilde{p}\, h^{\alpha\beta} + \Pi^{\alpha\beta}] \,. \tag{22}$$

Here, μ is the total energy density of matter measured by u^α, $\tilde{p}$ the isotropic pressure *, q^α the energy

* The notation is such that $p := \varrho\, F' - F$ is the isotropic pressure of General Relativity.

flux relative to u^α and $\Pi^{\alpha\beta}$ the anisotropic pressure [11]. With $h_\beta{}^\alpha := \delta_\beta{}^\alpha - c^{-2} u^\alpha u_\beta$ one finds:

$$\mu = c^{-2}(1-2\lambda F)^{-1}[F + 2\lambda \varkappa^{-1} h^{\alpha\beta} F_{,\alpha;\beta}], \quad (23\text{ a})$$

$$\begin{aligned}\tilde{p} = &(1-2\lambda F + 3\lambda \varrho F')^{-1} \\ &\cdot [\varrho F'(1-\lambda F)(1-2\lambda F)^{-1} - F] \\ &+ \tfrac{2}{3}\varkappa^{-1}\lambda(1-2\lambda F)^{-1} h^{\alpha\beta} F_{,\alpha;\beta} \\ &- 2\varkappa^{-1}\lambda(1-2\lambda F + 3\lambda\varrho F')^{-1} F_{,}{}^{\sigma}{}_{;\sigma},\end{aligned} \quad (23\text{ b})$$

$$q^\alpha = 2\lambda(1-2\lambda F)^{-1} h_\sigma{}^\alpha F_{,}{}^{\sigma}{}_{;\varkappa} u^\varkappa, \quad (23\text{ c})$$

$$\Pi^{\alpha\beta} = 2\lambda(1-2\lambda F)^{-1} h_\varkappa{}^\alpha h_\lambda{}^\beta \cdot (g^{\varkappa\sigma} g^{\lambda\tau} - \tfrac{1}{3} g^{\varkappa\lambda} g^{\sigma\tau}) F_{,\sigma;\tau}. \quad (23\text{ d})$$

It is checked easily that Eqs. (23 a – d) are obtained from Eq. (16) by putting $L_M = -F$ while Eqs. (20 b), (21) follow from Equation (14).

Due to the non-minimal coupling of matter and curvature the gravitating fluid no longer is an ideal fluid, in general. Also, the restriction $\mu + c^{-2}\tilde{p} > 0$ does not follow from $F + c^{-2} p > 0$. It seems obvious that the thermodynamics of a gravitating system in a theory without minimal coupling will be more complicated than in Einstein's theory. Fortunately, as shown below, for static and spherically symmetric (as well as for homogeneous and isotropic) matter sources $q^\alpha = \Pi^{\alpha\beta} = 0$.

4. Gravitating Point Particles

Often, as for example in kinetic theory, instead of the continuum description of matter the point particle approach is needed. In order to obtain, in the present theory, the equations of motion for N gravitating (monopole-) point particles of masses m_i whose worldlines are given by $a_i{}^\alpha = a_i{}^\alpha(\lambda_i)$ (λ_i an arbitrary parameter) we start from the Lagrangian:

$$\mathcal{L}_M = -\sum_{i=1}^{N} \int_{-\infty}^{+\infty} d\lambda_i \sqrt{-g}\, g_{\mu\nu}\, \hat{m}_i{}^{\mu\nu}(\lambda_i)\, \delta^4[x - a_i(\lambda_i)] \quad (24)$$

from which

$$T^{\alpha\beta} = \sum_{i=1}^{N} \int_{-\infty}^{+\infty} d\lambda_i\, m_i{}^{\alpha\beta}(\lambda_i)\, \delta^4[x - a_i(\lambda_i)] \quad (25)$$

with

$$m_i{}^{\alpha\beta} = \sqrt{-g}\, \hat{m}_i{}^{\alpha\beta}.$$

Now, from Eq. (14) written for a tensor density, by a well known method [12]

$$m_i{}^{\alpha\beta} = m_i \left(g_{\mu\nu} \frac{da_i{}^\mu}{d\lambda_i} \frac{da_i{}^\nu}{d\lambda_i} \right)^{-1/2} \quad (26)$$

where m_i are constants of integration. Introduction of proper time s_i in place of λ_i leads to the equations of motion **

$$\frac{D}{ds_i}(g_{\alpha\beta}\, a_i{}^\beta) = h_\alpha{}^\beta \frac{\partial}{\partial a_i{}^\beta}[\log(1 + \varepsilon l_0{}^2 R(a_i))]. \quad (27)$$

In Eq. (27) $g_{\alpha\beta}(a_i{}^\varkappa)$ is the gravitational field generated by all N particles taken at the position of the i-th particle. The equation of motion could have been obtained directly from Eq. (21) by letting $F'{}_{,}{}^\alpha \sim p_{,}{}^\alpha = 0$. Thus, in the present theory, gravitating monopole particles do not follow geodesics. Only for test particles this result of Einstein's theory is retained.

5. Newtonian Approximation

We want to know the conditions by which the theory of gravitating matter discussed here allows a Newtonian approximation. By calculating, for the metric

$$g_2{}^0 = [1 + 2c^{-2}\varphi(x^j)]\, c^2 dt^2 - [1 - 2c^{-2}\varphi(x^j)] \cdot [(dx^1)^2 + (dx^2)^2 + (dx^3)^2] \quad (28)$$

the l.h.s. and r.h.s. of the field equations (22) up to terms of order $\varphi/c^2 \cong v^2/c^2$ i.e. by assuming as usual that

$$\varphi/c^2 \cdot \varrho F' \ll F, \quad \varphi/c^2(\varrho F' - F) \ll \varrho F' - F$$

we arrive at the set of equations:

$$-(2/c^2)\Delta\varphi = (1-2\lambda F)^{-1}[-\varkappa F - 2\lambda F_{,}{}^{j}{}_{,j}]. \quad (29\text{ a})$$

$$0 = (1-2\lambda F)^{-1}(1-2\lambda F + 3\lambda\varrho F')^{-1} \cdot [-\varkappa(\varrho F' - F) + \lambda\varkappa(2F - \varrho F')], \quad (29\text{ b})$$

$$0 = -2\lambda(1-2\lambda F)^{-1} F_{,}{}^{i}{}_{,k} \text{ with } i \neq k. \quad (29\text{ c})$$

In order to obtain a meaningful result we must impose the condition:

$$l_0{}^2 F_{,i,k} \ll F. \quad (30)$$

Taking into account that, in lowest order, the pressure p is small as compared to the energy density we can interpret Eq. (30) as saying that the length D characteristical for the change of the density of rest mass $D = (\varrho/\Delta\varrho)^{1/2}$ is large with regard to l_0. With Eq. (16) $D \gg r_s$ which again is violated only for neutron stars and collapsing objects.

If, in addition to Eq. (30)

$$l_0{}^2 \varkappa F \ll 1 \quad (31)$$

** In this section, units have been chosen such that $c=1$.

then the Poisson equation $\Delta\varphi = 4\pi G\varrho$ is obtained. Equation (31) implies that the rest mass density ϱ of the gravitating body remains small as compared to the energy density $\varkappa^{-1} l_0^{-2} c^{-2}$. Putting the gravitational radius of objects in place of l_0 one concludes that for stars, white dwarfs and planets condition (31) is satisfied.

6. Static, Spherically Symmetric Gravitating Fluids

In order to investigate the consequences of the theory suggested in Sect. 2 for gravitational collapse * we study the case of gravitating fluids with a static and spherically symmetric gravitational field:

$$g_2{}^0 = e^{2\alpha(r)} c^2 \, dt^2 - e^{2\beta(r)} dr^2 - r^2 d\omega^2 . \qquad (32)$$

From Eq. (20 b) $F = F(r)$ immediately, while Eq. (21) can be integrated to give:

$$\alpha(r) + \log[F'(1+\varepsilon l_0^2 R^{\sigma}{}_{\cdot\sigma})] = \alpha_0 \qquad (33)$$

with α_0 a constant.

From Eq. (15), in this case:

$$R^{\sigma}{}_{\cdot\sigma} = (1-2\lambda F+3\lambda\varrho F')^{-1}\left\{4\varkappa F-3\varkappa\varrho F'-6\lambda e^{-2\beta}\left[\left(\alpha'-\beta'+\frac{2}{r}\right)\frac{dF}{dr}+\frac{d^2F}{dr^2}\right]\right\}. \qquad (34)$$

Combination of Eqs. (33) and (34) leads to a first order differential equation for either $\alpha(r)$ or $\beta(r)$, i.e.

$$e^{\alpha_0} = e^{\alpha} F'(1-2\lambda F+3\lambda\varrho F')^{-1}\left\{1+2\lambda F-6\lambda^2\varkappa^{-1}e^{-2\beta}\left[\left(\alpha'-\beta'+\frac{2}{r}\right)\frac{dF}{dr}+\frac{d^2F}{dr^2}\right]\right\}. \qquad (35)$$

The relevant field equations are **:

$$r^{-2}[r(e^{-2\beta}-1)]' = -\varkappa F(1-2\lambda F)^{-1}+2\lambda(1-2\lambda F)^{-1}e^{-2\beta}\left[\left(\frac{2}{r}-\beta'\right)\frac{dF}{dr}+\frac{d^2F}{dr^2}\right] \qquad (36)$$

and

$$r^{-2}(e^{-2\beta}-1)+2e^{-2\beta}\frac{\alpha'}{r} = (1-2\lambda F+3\lambda\varrho F')^{-1}(1-2\lambda F)^{-1} \qquad (37)$$

$$\cdot\left\{\varkappa\varrho F'(1-\lambda F)-\varkappa F(1-2\lambda F)+2\lambda e^{-2\beta}(1-2\lambda F)\left(\alpha'+\frac{2}{r}\right)\frac{dF}{dr}-6\lambda^2\varrho F' e^{-2\beta}\left(\frac{d^2F}{dr^2}-\beta'\frac{dF}{dr}\right)\right\}.$$

It seems not difficult, in principle, to extract $\alpha(r)$, $\beta(r)$ and $F(r)$ from these coupled equations. However, I have not yet been able to formulate the analogue of the Landau-Oppenheimer-Volkoff equation, in general.

In order to see qualitatively what will happen the crudest model possible is considered. This means taking constant rest mass density $\varrho = \varrho_0$. Due to $dF/dr = F'(d\varrho/dr) = 0$ the field equations reduce to

$$G_0{}^0 = r^{-2}[r(e^{-2\beta}-1)]' = -\varkappa F(1-2\lambda F)^{-1}, \qquad (38)$$

$$G_1{}^1 = G_2{}^2 = G_3{}^3 = r^{-2}(e^{-2\beta}-1)+2e^{-2\beta}\frac{\alpha'}{r}$$
$$= \varkappa(1-\lambda F+3\lambda\varrho_0 F')^{-1}(1-2\lambda F)^{-1} \cdot [\varrho_0 F'(1-\lambda F)-F(1-2\lambda F)]. \qquad (39)$$

Also $dR^{\sigma}{}_{\cdot\sigma}/dr = 0$. Thus Eq. (33), after differentiation, leads to the same equation as known from Einstein's theory:

$$\frac{dp}{dr} = -\frac{d\alpha}{dr}[p+\varrho_0(c^2+e(\varrho_0))]. \qquad (40)$$

Again, $p = \varrho_0^2(de/d\varrho_0)$ is the pressure as defined in General Relativity. Integration of Eq. (38) for $e^{-2\beta}$ and substitution of $e^{-2\beta}$ and $d\alpha/dr$ from Eq. (40) into Eq. (39) results in:

$$\frac{dp}{dr} = -\frac{r}{2L^2 u_0}\cdot\frac{1}{1-(r/L)^2} \cdot \frac{(u_0+p)[3p+u_0(1+4\lambda u_0)]}{1+\lambda u_0+3\lambda p} \qquad (41)$$

where

$$u_0 := \varrho_0[c^2+e(\varrho_0)], \quad L^{-2} := \frac{\varkappa}{3}u_0(1-2\lambda u_0)^{-1}.$$

In place of p the pressure $\tilde{p}$ as defined by the r.h.s. of Eq. (39) is introduced. With $z := \tilde{p}(1-2\lambda u_0)$ we obtain the equation

$$\frac{dz}{dr} = -\frac{r}{2L^2 u_0}\frac{1}{[1-(r/L)^2]} \cdot \frac{(1-\lambda u_0-3\lambda z)(u_0+z)(u_0+3z)}{1+2\lambda u_0} \qquad (42)$$

* The singularity theorems of Hawking and Penrose [13] cannot be applied in the theory without minimal coupling because the inequalities for the Ricci tensor used in the proof do not now follow from the energy inequalities.

** The prime on all quantities except F stands for d/dr.

whose integration leads to

$$\frac{1+z\,u_0{}^{-1}}{1+3\,z\,u_0{}^{-1}}\left(\frac{1-3\,\lambda\,z/(1-\lambda\,u_0)}{1+3\,z\,u_0{}^{-1}}\right)^{2\lambda u_0} = \left(\frac{1-a^2/L^2}{1-r^2/L^2}\right)^{1/2} \tag{43}$$

where $0 \leqq r \leqq a$.

At the edge of the star, i. e. for $r=a$, $\tilde{p}=0$ is assumed. For $\lambda=0$ the expression for $p(r)$ of the internal Schwarzschild solution is obtained. For $\lambda \neq 0$, in contrast to the situation in General Relativity, the central pressure $\tilde{p}(0)$ in the star remains finite if the signfactor ε occuring in λ is chosen to be positive and $\lambda\,u_0 < 1$ as was assumed. In fact, $\tilde{p}(r)$ remains bounded. This seems to indicate that gravitational collapse possibly can be avoided.

While the metric coefficient $e^{2\beta}$ is given by

$$e^{2\beta} = (1 - r^2/L^2)^{-1} \tag{44}$$

$e^{2\alpha}$ is determined implicitely through Eqs. (40) and (42). The exact solution thus obtained corresponds to the interior Schwarzschild solution.

7. Conclusion

The preceding sections were intended to show that a theory of gravitation can be constructed which closely mimics Einstein's theory but gives up the principle of minimal coupling. The theory suggested differs from General Relativity in the inside of matter, especially in regions of high density and density gradients. At this point of investigation, it looks as if the theory were free of conceptual and mathematical inconsistencies. Nevertheless, much further work must be done in order to make sure this claim (Cauchy initial problem, thermodynamics, kinetic theory, electromagnetism, post-Newtonian approximation, further exact solutions of the field equations, cosmology etc).

Two principal objections to the theory suggested here remain to be overcome: (1) the arbitrariness of the new constant l_0 of dimension length; (2) the ambiguity in the choice of the Lagrangian. Other possibilities than the one studied here are available. One may start, for example, from the scalar invariants $\overset{**}{R}{}^{\alpha\beta\gamma\delta} R_{\alpha\beta\gamma\delta}$ or $\overset{*}{R}{}^{\alpha\beta\gamma\delta} R_{\alpha\beta\gamma\delta}$ in place of $R^{\sigma}{}_{\sigma}$ in the matter part of the Lagrangian. Furthermore the coupling of matter variables and curvature can be done in a more geometrical manner. I shall report on such different approaches separately.

The study of gravitational collapse which was barely touched upon needs further elaboration. This is of special interest as many workers in the field seem to believe that gravitational collapse can be haltet solely by considering quantum phenomena. The breaking of minimal coupling suggests an approach to this problem in the framework of classical macroscopic field theory.

Gravitational theories without minimal coupling seem to be a very interesting subject for study if only for the reason of obtaining a better understanding of the exceptional role played by Einstein's theory.

1 L. P. Eisenhart, Riemannian Geometry, Princeton 1964.
2 A. Trautman, in Brandeis Summer Institute for Theoretical Physics 1964, Vol. 1, Englewood Cliffs, N. J. 1965, p. 124.
3 P. G. Roll, R. Krotkov, and R. H. Dicke, Ann. Phys. (N. Y.) **26,** 442 [1967].
4 V. B. Braginsky and V. I. Panov, GRG **3,** 403 [1972].
5 P. Rastall, Phys. Rev. **6** D, 3337 [1972].
6 W. Rindler, Essential Relativity, New York 1969, p. 197.
7 Ch. W. Misner, K. S. Thorne, and J. A. Wheeler, Gravitation, San Francisco 1973, p. 390.
8 J. Ehlers, Survey of General Relativity Theory, in Proceedings of the Banff Summer School 1972, Dordrecht 1973, p. 23.
9 J. Ehlers, General Relativity and Kinetic Theory, in Proceedings of the International School of Physics Enrico Fermi 1971, New York 1971.
10 J. R. Ray, J. Math. Phys. **13,** 1451 [1972].
11 G. F. R. Ellis, Relativistic Cosmology, in Proceedings of the International School of Physics Enrico Fermi 1971, New York 1971.
12 P. Havas and J. Goldberg, Phys. Rev. **128,** 398 [1962].
13 S. W. Hawking and G. F. R. Ellis, The large scale structure of space-time, Cambridge 1973.